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Type-I D-branes in an H-flux and twisted KO-theory

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Abstract: Witten has argued that charges of type-I D-branes in the presence of an H-flux, take values in twisted KO-theory. We begin with the study of real bundle gerbes and their holonomy. We then introduce the notion of real bundle gerbe KO-theory which we establish is a geometric realization of twisted KO-theory. We examine the relation with twisted K-theory, the Chern character and provide some examples. We conclude with some open problems.

Keywords: D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

It has been argued by Witten [1], that type-I D-branes charges take values in KO-theory, based on explicit calculations by [2] and [3]. Now the Neveu-Schwarz B-field in type I string theory is a local closed 2-form B on M which defines a class H in $H^2(M, \mathbb{Z}_2)$, as argued in [3] and the paragraphs after eq. (5.7) in Witten [1]. When the Neveu-Schwarz

B-field B is turned on, type-I D-branes can be wrapped on a submanifold Z of spacetime only if the following equation holds,

$$[H|_Z] + w_2(Z) = 0 \in H^2(Z, \mathbb{Z}_2)\,, \tag{1.1}$$

where $w_2(Z)$ denotes the second Stieffel-Whitney class of the tangent bundle of Z. Moreover, the normal bundle NZ of Z is KO-oriented if and only if it has a spin structure, i.e. $w_2(NZ) = 0$. However, this is not the case when there is a Neveu-Schwarz B-field if equation (1.1) is to be satisfied. This and other considerations enabled Witten in [1] to deduce that in the presence of a background flux $H \in H^2(M, \mathbb{Z}_2)$, type-I D-branes charges take values in *twisted* KO-theory, $KO(M, H)$. This forces one to work with twisted KO-theory. In this note, we first of all give a geometric interpretation of such 2-torsion B-fields as the holonomy of connections for real bundle gerbes, which is the real analog of the bundle gerbe theory in [4] and which was briefly discussed in [5]. Just as the holonomy of an orthogonal connection on a real line bundle represents the first Stieffel-Whitney class of the real line bundle, we establish here that the holonomy of an orthogonal connection on a real bundle gerbe represents its Dixmier-Douady class in $H^2(M, \mathbb{Z}_2)$, which is sometimes referred to in the physics literature as the t'Hooft class. We also give a geometric realization of twisted KO-theory as the KO-theory of real bundle gerbe modules, or briefly, real bundle gerbe KO-theory. We give both finite dimensional and infinite dimensional geometric realizations of twisted KO-theory, as both occur naturally in examples. This part can be viewed as the real analog of the paper [6]. Since dimension 10 is not crucial for our analysis, it is possible that it applies to more general orientifolds in string theory. As it was first shown in Gukov's paper [7], KO and KSp groups of spheres classify D-branes localized on orientifold planes of various dimension, of which type-I string theory is the simplest example. Therefore, one could expect that the twisted KO-groups discussed in our paper classify D-branes on orientifold planes in the presence of an H-flux, so that the dimensionality of the orientifold plane is related to the degree of suspension of the twisted KO-group. We thank Sergei Gukov for these comments.

This paper is organised as follows. Section 2 summarises the theory of real bundle gerbes. These are geometric objects that are associated with degree 2, $\mathbb{Z}_2$-valued Čech cohomology classes on M, and is the real analog of the paper by Murray [4]. The notion of stable equivalence of real bundle gerbes, which is essential for the understanding of the sense in which the degree 2 class (known as the Dixmier-Douady class of the real bundle gerbe) determines an associated bundle gerbe is the subject of section 3. Section 4 studies the Chern-Weil description of the Dixmier-Douady class of real bundle gerbes, using a generalization of holonomy. Here we encounter a problem that in a very small class of manifolds, the holonomy does not determine the Dixmier-Douady class of the real bundle gerbe, and we leave the geometric description of the Dixmier-Douady class of the real bundle gerbe for these manifolds as an open problem in the final section. Real bundle gerbe modules are introduced in section 5, and the KO-theory of real bundle gerbe modules in section 6. In section 7, we establish the equivalence between the real bundle gerbe KO-theory and the twisted KO-theory as defined in [8] and [9] in terms of real Azumaya algebra bundles. In section 8, an equivalent description of twisted KO-theory is given in

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terms of infinite dimensional real bundle gerbe modules with structure group $O_{\mathcal{K}}$, as well as other manifestations. In section 9, we briefly give the relation with the KO-theory of continuous trace real C^*-algebras. Sections 5–9 can be viewed as the real analog of the paper by Bouwknegt, Carey, Mathai, Murray and Stevenson [6]. In section 10, we define the complexification map, which is a homomorphism from $KO(M, H)$ to $K(M, \beta(H))$, where β is the Bockstein homomorphism. The complexification map complexifies the real bundle gerbe into a complex bundle gerbe, as well as real bundle gerbe modules into bundle gerbe modules. This enables us to define the twisted Chern character from KO-theory to cohomology and we establish that all the components of degree $(4n - 2)$ vanish for any positive integer n. We also sketch the spectral sequence relevant for the calculation of twisted KO-groups and give examples. We end with some concluding remarks and open problems in section 11.

2. Real bundle gerbes

2.1 Real bundle gerbes and Dixmier-Douady classes

Here we define real bundle gerbes in analogy with the definition given in [4] in the complex case. We refer to [4, 6] for standard notation regarding fibre products etc. Recall that an orthogonal line bundle $L \to M$ is a real line bundle with a fibrewise orthogonal inner product. For such a real line bundle the set of all vectors of norm 1 is a principal $\mathbb{Z}_2$ bundle. Conversely if $P \to M$ is a principal $\mathbb{Z}_2$ bundle then associated to it is a real line bundle with fibrewise orthogonal inner product. This is formed in the standard way as the quotient of $P \times \mathbb{R}$ by the action of $\mathbb{Z}_2$ given by $(p, z)w = (pw, w^{-1}z)$ where $w \in \mathbb{Z}_2$. The theory of real bundle gerbes can use either principal $\mathbb{Z}_2$ bundles or equivalently orthogonal line bundles. In the discussion below we will mostly adopt the latter point of view. All maps between orthogonal line bundles will be assumed to preserve the inner product unless we explicitly comment otherwise.

A real bundle gerbe over M is a pair (L, Y) where $\pi\colon Y \to M$ is a submersion and L is an orthogonal line bundle $L \to Y^{[2]}$ on the fibre product $Y^{[2]}$ which has a product, that is, an isometric isomorphism

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)} \tag{2.1}$$

for every (y_1, y_2) and (y_2, y_3) in $Y^{[2]}$. We require the product to be smooth in y_1, y_2 and y_3 but in the interests of brevity we will not state the various definitions needed to make this requirement precise, they can be found in [4]. The product is required to be associative whenever triple products are defined. Also in [4] it is shown that the existence of the product and the associativity imply isomorphisms $L_{(y,y)} \simeq \mathbb{R}$ and $L_{(y_1,y_2)} \simeq L^*_{(y_2,y_1)}$. We shall often refer to a real bundle gerbe (L, Y) as just L. If the bundle gerbe (L, Y) has Y finite-dimensional we will say that it is a *finite-dimensional* bundle gerbe.

One defines such notions as morphisms of real bundle gerbes, products and duals exactly as one does for the case of complex bundle gerbes. It is perhaps worth emphasising the point that the product of two real bundle gerbes (L, Y) and (J, X) is formed over the fibre product $Y \times_M X$ so that $L \otimes J$ is the real line bundle over $(Y \times_M X)^{[2]}$ whose fibre at a point $((y_1, x_1), (y_2, x_2))$ is $L_{(y_1,y_2)} \otimes J_{(x_1,x_2)}$. If J is a orthogonal line bundle over Y then we can define a real bundle gerbe $\delta(J)$ by $\delta(J) = \pi_1^*(J) \otimes \pi_2^*(J)^*$, that is $\delta(J)_{(y_1,y_2)} = J_{y_2} \otimes J_{y_1}^*$, where $\pi_i : Y^{[2]} \to Y$ is the map which omits the ith element. The product on $\delta(J)$ is induced by the natural pairing

$$J_{y_2} \otimes J_{y_1}^* \otimes J_{y_3} \otimes J_{y_2}^* \to J_{y_3} \otimes J_{y_1}^* .$$

A real bundle gerbe which is isomorphic to a real bundle gerbe of the form $\delta(J)$ is called *trivial*. A choice of J and a real bundle gerbe isomorphism $\delta(J) \simeq L$ is called a *trivialisation*. As is the case for complex bundle gerbes, any two trivialisations J and K of a real bundle gerbe L differ by the pullback of a real line bundle on M. Hence the set of all trivialisations of a given real bundle gerbe is naturally acted on by the set of all orthogonal line bundles on M. This is analogous to the way in which the set of all trivialisations of an orthogonal line bundle $L \to M$ is acted on by $\mathbb{Z}_2$.

One can think of real bundle gerbes as one stage in a hierarchy of objects with each type of object having a characteristic class in $H^p(M, \mathbb{Z}_2)$. For example if $p = 1$ we have real line bundles on M, the characteristic class is the first Stieffel-Whitney class. When $p = 2$ we have real bundle gerbes and they have a characteristic class $d(L) = d(L, Y) \in H^2(M, \mathbb{Z}_2)$, called the Dixmier-Douady class of (L, Y), and is the obstruction to the gerbe being trivial. The class $d(L)$ is constructed in essentially the same way as in [4]. Let $P \to Y^{[2]}$ be the orthogonal frame bundle associated to the orthogonal line bundle L. Choose an open cover $\{U_i\}_{i \in I}$ of M such that there exist local sections $s_i: U_i \to Y$ of $\pi: Y \to M$ and such that each non-empty intersection $U_{i_1} \cap \cdots \cap U_{i_p}$ is contractible. Form maps $(s_i, s_j): U_{ij} \to Y^{[2]}$ in the usual way by sending $m \in U_{ij} = U_i \cap U_j$ to $(s_i(m), s_j(m)) \in Y^{[2]}$. Let P_{ij} denote the pullback bundle on $Y^{[2]}$. The product on the bundle gerbe L induces one on the principal $\mathbb{Z}_2$ bundle P and hence we have isomorphisms $P_{ij} \otimes P_{jk} \to P_{ik}$. Choose sections σ_{ij} of P_{ij} and define $\epsilon_{ijk}: U_{ijk} \to \mathbb{Z}_2$ by $\sigma_{ij}\sigma_{jk} = \sigma_{ik}\epsilon_{ijk}$. The associativity of the bundle gerbe product ensures that ϵ_{ijk} satisfies the Čech 2-cocycle condition $\epsilon_{jkl}\epsilon_{ikl}^{-1}\epsilon_{ijl}\epsilon_{ijk}^{-1} = 1$.

Theorem 1. *A real bundle gerbe (L, Y) has zero Dixmier-Douady class precisely when it is trivial.*

The proof is the obvious adaption of that in [4]. If the bundle is trivial we can make the sections in the discussion above global and thus obtain a trivial class. If the class is trivial then we can find h_{ij} such that $(h_{ij}\sigma_{ij})(h_{ik}\sigma_{ik})^{-1}(h_{jk}\sigma_{jk}) = 1$. We let $Y_i = \pi^{-1}(U_i)$ and define a bundle Q_i by $(Q_i)_y = P_{(s_i(\pi(y)),y)}$. Over $Y_i \cap Y_j$ the bundle gerbe multiplication and the $h_{ij}\sigma_{ij}$ define clutching isomorphisms $\hat{\sigma}_{ij}: Y_i \to Y_j$ which satisfy $\sigma_{ij}\sigma_{ik}^{-1}\sigma_{jk} = 1$ and hence glue the Q_i together to define a global bundle $Q \to Y$. It is straightforward to show that this is a trivialisation of P.

Notice that the same is true of the other objects in our hierarchy, line bundles are trivial if and only if their Stieffel-Whitney class vanishes. The Dixmier-Douady class behaves as one would expect, it is additive under forming products, it changes sign when we take duals, and it is natural with respect to pullbacks.

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2.2 Lifting real bundle gerbes, pfaffian line bundles and surjectivity

We will need one type of example of a real bundle gerbe in a number of places. Consider a central extension of groups

$$\mathbb{Z}_2 \to \hat{G} \to G\,.$$

If $Y \to M$ is a principal G bundle then it is well known that the obstruction to lifting Y to a $\hat{G}$ bundle $\hat{Y}$ that covers Y is a class in $H^2(M, \mathbb{Z}_2)$. It will be shown that a real bundle gerbe can be constructed from Y, the so-called lifting real bundle gerbe, whose Dixmier-Douady class is the obstruction to lifting Y to a $\hat{G}$ bundle. The construction of the lifting real bundle gerbe is quite simple. As Y is a principal bundle there is a map $g\colon Y^{[2]} \to G$ defined by $y_1 g(y_1, y_2) = y_2$. We use this to pull back the $\mathbb{Z}_2$ bundle $\hat{G} \to G$ and form the associated orthogonal line bundle $L \to Y^{[2]}$. The real bundle gerbe product is induced by the group structure of $\hat{G}$.

Some finite dimensional examples of lifting real bundle gerbes that we will be interested in are those associated to the central extensions,

$$\mathbb{Z}_2 \to \mathrm{SO}(n) \to PO(n)\,, \qquad \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n)\,.$$

It is well known that the obstruction to lifting a principal $\mathrm{SO}(n)$ bundle Y to a $\mathrm{Spin}(n)$ bundle is the second Stieffel-Whitney class $w_2(Y) \in H^2(M, \mathbb{Z}_2)$, which is also equal to the second Stieffel-Whitney class of the associated orthogonal vector bundle $Y \times_{\mathrm{SO}(n)} \mathbb{R}^n$. In the physics literature, the real line bundle on $\mathrm{SO}(n)$ that is associated to the $\mathrm{Spin}(n)$ double cover is known as the pfaffian line bundle Pfaff. It is occurs as a topological anomaly as follows. Real vector bundles E of rank n over S^1 are determined by an element $g \in \mathrm{SO}(n)$ by the clutching construction. So the self-adjoint family of Dirac operators $\not{\partial}_{S \otimes E_g}$ on the real vector bundles $S \otimes E_g$, where S is the non-trivial real bundle of spinors on S^1, is parametrized by the group $\mathrm{SO}(n)$. Freed and Witten [10] explained that the real determinant line bundle of this family, which is essentially defined as the highest exterior power of the kernel of the family, defines a non-trivial real line bundle Pfaff on the group $\mathrm{SO}(n)$, which has the property that its lift to $\mathrm{Spin}(n)$ is the trivial real line bundle. They also determine a connection and holonomy of the pfaffian line bundle.

Another interesting $\mathbb{Z}_2$ central extension that occurs in physics is

$$\mathbb{Z}_2 \to Mp(n) \to \mathrm{Sp}(n)\,,$$

where $\mathrm{Sp}(n)$ denotes the symplectic group in $\mathbb{R}^{2n}$ and $Mp(n)$ the metaplectic group. It is well known that the obstruction to lifting a principal $\mathrm{Sp}(n)$ bundle Y to a $Mp(n)$ bundle is the second Stieffel-Whitney class $w_2(Y) \in H^2(M, \mathbb{Z}_2)$, which is also equal to the second Stieffel-Whitney class of the associated symplectic vector bundle $Y \times_{\mathrm{Sp}(n)} \mathbb{R}^{2n}$.

An infinite dimensional $\mathbb{Z}_2$ central extension that occurs in physics is

$$\mathbb{Z}_2 \to \mathrm{SO}(\mathcal{H}) \to PO(\mathcal{H})\,,$$

for $\mathcal{H}$ an infinite dimensional, separable, real Hilbert space. By Kuiper's theorem, one knows that $\mathrm{SO}(\mathcal{H})$ is a contractible group. Therefore $PO(\mathcal{H})$ is a model for the Eilenberg-Maclane space $K(\mathbb{Z}_2, 1)$, and so $H^2(X, \mathbb{Z}_2) = [X, BPO(\mathcal{H})]$. Therefore,

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Theorem 2. *The lifting real bundle gerbes of principal $PO(\mathcal{H})$ bundles over X are classified up to isomorphism by their Dixmier-Douady invariant in $H^2(X, \mathbb{Z}_2)$. Moreover, every element in $H^2(X, \mathbb{Z}_2)$ determines a lifting real bundle gerbe of a principal $PO(\mathcal{H})$ bundle over X, up to isomorphism.*

The following examples of real bundle gerbes follow the analogous construction in [11]. Let M be a compact Riemann surface, and choose a point $p \in M$. Now cover M with two open sets, $U_1 \cong \mathbb{R}^2$ a coordinate neighbourhood of p, and $U_0 = M \setminus \{p\}$. Then

$$U_0 \cap U_1 \cong \mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R}.$$

We define a real bundle gerbe $\mathcal{G}_p$ as follows. Let Y be the disjoint union of U_0 and U_1; then there is a submersion $Y \to M$. Consider the real line bundle $L = L_{01}$ on $U_0 \cap U_1$ which is the pull-back from S^1 of the real line bundle whose first Stieffel-Whitney class is the generator of $H^1(S^1, \mathbb{Z}_2)$ and whose associated principal bundle is the non-trivial double cover of S^1. The choice of orientation on M gives an orientation on S^1 and hence a choice of generator. It is easy to see that $L \to Y^{[2]}$ defines a bundle gerbe. Since M is compact, the Dixmier-Douady class of $\mathcal{G}_p$ is the generator of $H^2(M, \mathbb{Z}_2) \cong \mathbb{Z}_2$. More generally, consider an oriented codimension 2 submanifold Q of a compact oriented manifold M. The previous construction can be routinely generalized to this situation. Take coordinate neighbourhoods U_α of M along Q, then $U_\alpha \cong (U_\alpha \cap Q) \times \mathbb{R}^2$. Now take $U_0 = X \backslash N(Q)$, where $N(Q)$ is the closure of a sufficiently small tubular neighbourhood of Q that is diffeomorphic to the disc bundle in the normal bundle. Then

$$U_0 \cap U_\alpha \cong U_\alpha \cap Q \times \{x \in \mathbb{R}^2 : \|x\| > \epsilon\}$$

and as before we define Y to be the disjoint union of the U_α and U_0 so that we have a submersion $Y \to M$. Define the real bundle gerbe $L \to Y^{[2]}$ as follows. Let $L_{\alpha 0}$ be the pull-back by $x \mapsto x/\|x\|$ of the real line bundle with non-trivial Stieffel-Whitney class on S^1. The real line bundle $L_{\alpha\beta} = L_{\alpha 0} L_{\beta 0}^{-1}$ is defined on

$$U_\alpha \cap U_\beta \backslash N(Q) \cong (U_\alpha \cap U_\beta \cap Q) \times \{x \in \mathbb{R}^2 : \|x\| > \epsilon\} \simeq S^1 .$$

But by construction $w_1(L_{\alpha\beta}) = 0$ is zero on S^1 and so extends to a trivial real line bundle on the whole of $U_\alpha \cap U_\beta$. This defines real line bundles on overlaps, and hence the real line bundle $L \to Y^{[2]}$. It is not hard to see that the real bundle gerbe property is satisfied, hence it defines a real bundle gerbe $\mathcal{G}_Q$ on M that is associated to Q.

3. Stable isomorphism of real bundle gerbes

There are many real bundle gerbes which have the same Dixmier-Douady class but which are not isomorphic. We can define a notion of *stable isomorphism* for real bundle gerbes in exactly the same way as we can for complex bundle gerbes. Just as is the case for complex bundle gerbes, two real bundle gerbes have the same Dixmier-Douady class precisely when there is a stable isomorphism between them.

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Definition 1. *A stable isomorphism between real bundle gerbes* (L, Y) *and* (J, Z) *is a trivialisation of* $L^* \otimes J$*.*

The proof of the following proposition is entirely analogous to the proof for complex bundle gerbes.

Proposition 1. *A stable isomorphism exists from* (L, Y) *to* (J, Z) *if and only if* $d(L) = d(J)$*.*

If a stable isomorphism exists from (L, Y) to (J, Z) we say that (L, Y) and (J, Z) are stably isomorphic. It follows easily that stable isomorphism is an equivalence relation. Theorem 2 establishes that every class in $H^2(M, \mathbb{Z}_2)$ is the Dixmier-Douady class of some real bundle gerbe. Hence we can deduce from proposition 1 the

Theorem 3. *The Dixmier-Douady class defines a bijection between stable isomorphism classes of real bundle gerbes and* $H^2(M, \mathbb{Z}_2)$*.*

Stable isomorphisms between real bundle gerbes can be composed in exactly the same way as stable isomorphisms for complex bundle gerbes. We refer to [6] for more details. Following Serre, Patterson [12] established that every element in $H^2(X, \mathbb{Z}_2)$ determines a principal $PO(n)$ bundle over X. By the construction above this determines a lifting bundle gerbe and combining this with theorem 3, one has,

Theorem 4. *The Dixmier-Douady class defines a bijection between stable isomorphism classes of finite dimensional real bundle gerbes and* $H^2(M, \mathbb{Z}_2)$*.*

4. Holonomy and Chern Weil theory of real Dixmier-Douady classes

A U(1) bundle gerbe with connection and curving determines a Deligne cohomology class. Interpreted as a Cheeger-Simons differential character this is a pair consisting of a homomorphism $h\colon Z_2(M) \to \mathrm{U}(1)$ and a three-form ω on M satisfying

$$h(\partial\sigma) = \exp\left(\int_\sigma \omega\right)$$

for any three-cycle σ. The map h applied to smooth surfaces is just the holonomy and ω is the three-curvature. Observe that $H_2(M, \mathbb{Z})$ is generated by fundamental strings, which are just smooth maps of surfaces into M. h will be the holonomy for a real $\mathbb{Z}_2$ bundle gerbe, which takes its values in $\mathbb{Z}_2$.

For a real $\mathbb{Z}_2$ bundle gerbe the connection is flat and its curvature is zero. We can therefore take as a curving just the zero two-form on Y and it follows that the three-curvature on M is zero. In such a case h applied to boundaries is one so that it factors to a homomorphism $h\colon H_2(M, \mathbb{Z}) \to \mathbb{Z}_2$.

On the other hand for a real $\mathbb{Z}_2$ bundle gerbe we have the Dixmier-Douady class in $H^2(M, \mathbb{Z}_2)$ which we can map, as in the universal coefficient theorem cf. (4.1), to $\mathrm{Hom}(H_2(M, \mathbb{Z}), \mathbb{Z}_2)$.

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We wish to show that this map is just the holonomy. As we are assuming that $H_2(M, \mathbb{Z})$ is generated by fundamental strings, it suffices to prove this for $M = \Sigma$ a smooth surface.

Let Σ then be a closed surface. Choose an open cover U_0, U_1 where U_0 is a neighbourhood of a point in Σ and U_1 is a slightly enlarged open neighbourhood of the complement of U_0 chosen so that $U_0 \cap U_1$ is an annular region.

Consider a $\mathbb{Z}_2$ real bundle gerbe (P, Y) over Σ. This classified by a class in $H^2(\Sigma, \mathbb{Z}_2) = \mathbb{Z}_2$. Let $Y_i = \pi^{-1}(U_i)$. As $H^2(U_i, \mathbb{Z}_2) = 0$ the restriction of (P, Y) to U_0 and U_1 is trivial. So we can find a $\mathbb{Z}_2$ bundles $Q_i \to Y_i$ such that $\delta(Q_i) = P$ for each $i = 0, 1$. Denote by Q_{01} the $\mathbb{Z}_2$ bundle over $U_0 \cap U_1$ which is the difference between Q_0 and Q_1. If we put on (P, Y) a flat connection then this is inherited by Q_{01} and its holonomy is either $+1$ if Q_{01} is trivial or -1 if Q_{01} is not trivial. The original bundle gerbe (P, Y) is stably isomorphic to its pullback to the disjoint union of the two open sets U_0 and U_1 and hence is trivial if and only if we can find $\mathbb{Z}_2$ bundles $T_i \to U_i$ with $T_0 = Q_{01} \otimes T_1$ on $U_0 \cap U_1$. As U_0 is a disk we have that T_0 is trivial. Also from Mayer-Vietoris we have that $H^1(U_1, \mathbb{Z}_2) = H^1(\Sigma, \mathbb{Z}_2)$ so that T_1 extends to all of Σ, hence restricted to $U_0 \cap U_1$ it is trivial. Thus Q_{01} is trivial. We conclude that (P, Y) is trivial if and only if Q_{01} is trivial and hence the class in $H^2(\Sigma, \mathbb{Z}_2) = \mathbb{Z}_2$ is the same as the class determined by Q_{01} in $H^1(U_0 \cap U_1, \mathbb{Z}_2) = \mathbb{Z}_2$.

Now we calculate the holonomy. Because $H^3(\Sigma, \mathbb{Z}) = 0$ we can find a U(1) bundle $R \to Y$ such that $\delta(R) = P \times_{\mathbb{Z}_2} \mathrm{U}(1)$. Denote by R_i the U(1) bundle over U_i which is the difference between the trivialisations $Q_i \times_{\mathbb{Z}_2} \mathrm{U}(1)$ and R. We have $\pi^{-1}(R_0) \otimes Q_0 = R = \pi^{-1}(R_1) \otimes Q_1$ and $Q_0 = \pi^{-1}(Q_{01}) \otimes Q_1$ and hence on $U_0 \cap U_1$ we must have

$$R_1 = Q_{01} \otimes R_0 \,.$$

General bundle gerbe theory tells us that we can choose a connection on R which under δ is equal to the connection on Q. This means that we can choose connections D_i on R_i such that on $U_0 \cap U_1$ we have

$$D_1 = D + D_0$$

where D is the flat connection on Q_{01}. We claim that the holonomy of the connection and curving of (P, Y) is the holonomy of D around any non-trivial loop in $U_0 \cap U_1$. To see this note first that F_0, the curvature of D_0, is a closed two-form on a disk so we have $F_0 = d\alpha_0$ and hence $D_0 - \alpha_0$ is flat. If we restrict α_0 to $U_0 \cap U_1$ we can extend it to α_1 on $U_0 \cap U_1$. Letting $\tilde{D}_i = D_i - \alpha_i$ we note that we still have

$$\tilde{D}_1 = D + \tilde{D}_0$$

but now $\tilde{D}_0$ is flat. The holonomy of the bundle gerbe is the exponential of the integral of the global two-form F which is F_1 on U_1 and F_0 on U_0 but $F_0 = 0$ so the integral is just over U_1. Hence the holonomy of the bundle gerbe is the holonomy of D_1 over $U_0 \cap U_1$ but that is the product of the holonomy of D and the holonomy of D_0. The latter is the identity as $F_0 = 0$. So we have the required result.

These results apply also to lifting bundle gerbes which for SO(3) bundles we can also see directly. Let $P \to \Sigma$ be a principal SO(3) bundle. This is classified by the class in $H^2(\Sigma, \mathbb{Z}_2) = \mathbb{Z}_2$ which is the obstruction to lifting to SU(2). Indeed if P is trivial it lifts

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to SU(2) and the class vanishes. If P is not trivial and it lifts to an SU(2) bundle $\hat{P}$ then, as SU(2) is simply-connected $\hat{P}$ is trivial so that $P = \hat{P}/\mathbb{Z}_2$ is trivial. Hence P is trivial if and only if the class in $H^2(\Sigma, \mathbb{Z}_2)$ vanishes. This also implies that $P \to \Sigma$ is trivial on restriction to U_0 and U_1 and $H^2(U_i, \mathbb{Z}_2) = 0$ so there is a clutching map $U_0 \cap U_1 \to \mathrm{SO}(3)$ and, as $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$ there are two cases, either the map is homotopic to a constant and P is trivial or the map is not homotopic to a constant and P is not trivial. Equivalently the clutching map defines a class in $H^1(U_0 \cap U_0, \mathbb{Z}_2) = \mathbb{Z}_2$ which vanishes precisely when P is trivial i.e as an element of $\mathbb{Z}_2$ it is the same as the class in $H^2(\Sigma, \mathbb{Z}_2)$.

One can ask the question as to what extent does the holonomy of the real bundle gerbe determine the bundle gerbe? This is revealed by the universal coefficient theorem, which in this case is the split exact sequence

$$0 \to \mathrm{Ext}(H_1(M, \mathbb{Z}), \mathbb{Z}_2) \to H^2(\Sigma, \mathbb{Z}_2) \to \mathrm{Hom}(H_2(M, \mathbb{Z}), \mathbb{Z}_2) \to 0\,. \tag{4.1}$$

It is well known that $\mathrm{Ext}(H_1(M, \mathbb{Z}), \mathbb{Z}_2) = 0$ whenever the torsion subgroup of $H_1(M, \mathbb{Z})$ has only odd order torsion subgroups, cf. [13, page 195].

5. Real bundle gerbe modules

We can also define modules for real bundle gerbes in an almost identical fashion to the complex case. Thus let (L, Y) be a real bundle gerbe over a manifold M and let $E \to Y$ be a finite rank, orthogonal vector bundle. Assume that there is a vector bundle isomorphism

$$\phi\colon L \otimes \pi_1^* E \xrightarrow{\sim} \pi_2^* E \tag{5.1}$$

which is compatible with the real bundle gerbe multiplication in the obvious sense. In such a case we call E a real bundle gerbe module and say that the real bundle gerbe acts on E.

We define two real bundle gerbe modules to be isomorphic if they are isomorphic as real vector bundles and the isomorphism preserves the action of the real bundle gerbe. Denote by $\mathrm{Mod}_{\mathbb{R}}(L)$ the set of all isomorphism classes of real bundle gerbe modules for L. If (L, Y) acts on E and also on F then it acts on $E \oplus F$ in the obvious diagonal manner. The set $\mathrm{Mod}_{\mathbb{R}}(L)$ is therefore a semi-group. Note that if E is a real bundle gerbe module, then it is of rank a multiple of $2^{[j/2]}$.

Suppose the real bundle gerbe L has a trivialisation K. Then we have fibre-wise isomorphisms $L_{(y_1, y_2)} = K_{y_2} \otimes K^*_{y_1}$ for points y_1 and y_2 of Y lying in the same fibre over M. Therefore if E is an (L, Y) real bundle gerbe module then we have fibre-wise isomorphisms $K_{y_2} \otimes E_{y_2} \simeq K_{y_1} \otimes E_{y_1}$. In fact it can be shown (see [6]) that the bundle $K \otimes E$ descends to a real vector bundle on M. Conversely if F is a real vector bundle on M then L acts on $K \otimes \pi^*(F)$. Denote by $\mathrm{Bun}_{\mathbb{R}}(\mathrm{M})$ the semi-group of all isomorphism classes of real vector bundles on M. Then we have. we have

Proposition 2. *A trivialisation of* (L, Y) *defines a semi-group isomorphism from* $\mathrm{Mod}_{\mathbb{R}}(L)$ *to* $\mathrm{Bun}_{\mathbb{R}}(\mathrm{M})$.

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Notice that this isomorphism is not canonical but depends on the choice of the trivialisation. If we change the trivialisation by tensoring with the pull-back of a real line bundle J on M then the isomorphism changes by composition with the endomorphism of $\mathrm{Bun}_{\mathbb{R}}(\mathrm{M})$ defined by tensoring with J.

Recall that a stable isomorphism from a real bundle gerbe (L, Y) to a real bundle gerbe (J, X) is a trivialisation of $L^* \otimes J$. This means there is a real line bundle $K \to Y \times_f X$ and an isomorphism $L^* \otimes J \to \delta(K)$ or, in other words, for every (y_1, y_2) and (x_1, x_2) we have an isomorphism

$$L^*_{(y_1,y_2)} \otimes J_{(x_1,x_2)} \to K_{(y_2,x_2)} \otimes K^*_{(y_1,x_1)} .$$

Let $E \to Y$ be a real bundle gerbe module for L and define $\hat{F}_{(y,x)} = K^*_{(y,x)} \otimes E_y$ a bundle on $Y \times_f X$. We have isomorphisms

$$\begin{aligned} \hat{F}_{(y_2,x)} &= K^*_{(y_2,x)} \otimes E_{y_2} \\ &= K^*_{(y_1,x)} \otimes L_{(y_1,y_2)} \otimes E_{y_2} \\ &= K^*_{(y_1,x)} \otimes E_{y_1} \\ &= \hat{F}_{(y_1,x)} . \end{aligned}$$

It is not hard to see that the bundle $\hat{F}$ must in fact be the pullback under the map $Y \times_f X \to X$ of an orthogonal bundle F on X. The bundle F on X is in fact a (J, X) module since we have isomorphisms

$$\begin{aligned} J_{(x_1,x_2)} \otimes F_{x_2} &= J_{(x_1,x_2)} \otimes K^*_{(y,x_2)} \otimes E_y \\ &= K^*_{(y,x_1)} \otimes E_y \\ &= F_{x_1} \end{aligned}$$

Therefore a choice of stable isomorphism defines a map

$$\mathrm{Mod}_{\mathbb{R}}(L) \to \mathrm{Mod}_{\mathbb{R}}(J) .$$

In a similar fashion we can define an inverse map

$$\mathrm{Mod}_{\mathbb{R}}(J) \to \mathrm{Mod}_{\mathbb{R}}(L) .$$

Hence we have the following real analogue of proposition 4.3 of [6].

Proposition 3. *A stable isomorphism from (L, Y) to (J, Y) induces an isomorphism of semi-groups between $\mathrm{Mod}_{\mathbb{R}}(L)$ and $\mathrm{Mod}_{\mathbb{R}}(J)$.*

Note that, as in the trivial case, this isomorphism is not canonical but depends on the choice of stable isomorphism. Changing the stable isomorphism by tensoring with the pull-back of a real line bundle J over M changes the isomorphism in proposition 3 by composition with the endomorphism of $\mathrm{Mod}_{\mathbb{R}}(J)$ induced by tensoring with the pull-back of J.

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There is a close relationship between real bundle gerbe modules and bundles of real projective spaces. Recall that a bundle of real projective spaces $\mathcal{P} \to M$ is a locally trivial fibration whose fibres are isomorphic to $P(V)$ for V a real Hilbert space, either finite or infinite dimensional, and whose structure group is $PO(V)$. This means that there is a $PO(V)$ bundle $X \to M$ and $\mathcal{P} = X \times_{PO(V)} P(V)$. Associated to X is a lifting real bundle gerbe $J \to X^{[2]}$ and a Dixmier-Douady class. This Dixmier-Douady class is the obstruction to $\mathcal{P}$ being the projectivisation of a real vector bundle. The lifting real bundle gerbe acts naturally on the real bundle gerbe module $E = X \times H$ because each $J_{(x_1,x_2)} \subset O(V)$ by construction 2.2.

Let (L, Y) be a real bundle gerbe and $E \to Y$ a real bundle gerbe module. Then the projectivisation of E descends to a real projective bundle $\mathcal{P}_E \to M$ because of the real bundle gerbe action. It is straightforward to check that the class of this real projective bundle is $d(L)$. Conversely if $\mathcal{P} \to M$ is a real projective bundle with class $d(L)$ the associated lifting real bundle gerbe has class $d(L)$ and hence is stably isomorphic to (L, Y). So the module on which the lifting real bundle gerbe acts defines a module on which (L, Y) acts. From the discussion before 3 one can see that if two modules are related by a stable isomorphism they give rise to the same real projective bundle on M. We also have that $E \to Y$ and $F \to Y$ give rise to isomorphic projective bundles on M if and only if there is a line bundle $K \to M$ with $E = \pi^*(K) \otimes F$. Denote by $\mathrm{Lin}_{\mathbb{R}}(M)$ the group of all isomorphism classes of real line bundles on M. Then this acts on $\mathrm{Mod}_{\mathbb{R}}(L)$ by $E \mapsto \pi^*(K) \otimes E$ for any real line bundle $K \in \mathrm{Lin}_{\mathbb{R}}(M)$. If $[H] \in H^2(M, \mathbb{Z}_2)$, denote by $\mathrm{Pro}_{\mathbb{R}}(M, [H])$ the set of all isomorphism classes of real projective bundles with class $[H]$. Then we have

Proposition 4. *If (L, Y) is a real bundle gerbe then the map which associates to any element of $\mathrm{Mod}_{\mathbb{R}}(L)$ a real projective bundle on M whose Dixmier-Douady class is equal to $d(L)$ induces a bijection*

$$\frac{\mathrm{Mod}_{\mathbb{R}}(L)}{\mathrm{Lin}_{\mathbb{R}}(M)} \to \mathrm{Pro}_{\mathbb{R}}(M, d(L)) .$$

We next give some examples of real bundle gerbe modules. The first example involves Clifford algebras and spinors. Suppose that P is a principal $\mathrm{SO}(n)$ bundle that does not have a $\mathrm{Spin}(n)$-structure i.e. $w_2(P) \neq 0$, for example, P can be the principal bundle of oriented frames on a manifold M with $w_2(M) \neq 0$. We can consider the lifting real bundle gerbe and obtain a principal $\mathbb{Z}_2$ bundle $L \to P^{[2]}$ (or equivalently a real line bundle over $P^{[2]}$). It is natural to call this the Spin real bundle gerbe. The Dixmier-Douady invariant of the Spin real bundle gerbe coincides with the second Stieffel-Whitney class of P in $H^2(M, \mathbb{Z}_2)$. The pullback π^*P of P to P has a lifting to a $\mathrm{Spin}(n)$ bundle $\widehat{\pi^*P} \to P$. We consider the associated bundle of spinors by taking an irreducible spin representation V of $\mathrm{Spin}(n)$ and forming the associated real vector bundle $\mathcal{S} = \widehat{\pi^*P} \times_{\mathrm{Spin}(n)} V$ on P. $\mathcal{S}$ is a bundle gerbe module for L, called a spinor real bundle gerbe module. It is not hard to show that the possible spinor bundle gerbe modules for the Spin bundle gerbe $L \to P^{[2]}$ are parametrised by $H^1(M, \mathbb{Z}_2)$, i.e. the real line bundles on M, by following closely the proof given above in the $\mathrm{Spin}^{\mathbb{C}}$ case in [5].

The second example involves Weyl algebras and real symplectic spinors. Suppose that P is a principal $\mathrm{Sp}(n)$ bundle that does not have a $Mp(n)$-structure i.e. $w_2(P) \neq 0$, for example, for example, P can be the principal bundle of symplectic frames on a symplectic manifold M with $w_2(M) \neq 0$. We can consider the lifting real bundle gerbe and obtain a principal $\mathbb{Z}_2$ bundle $L \to P^{[2]}$ (or equivalently a real line bundle over $P^{[2]}$). It is natural to call this the metaplectic real bundle gerbe. The Dixmier-Douady invariant of this metaplectic real bundle gerbe coincides with the second Stieffel-Whitney class of P in $H^2(M, \mathbb{Z}_2)$. The pullback π^*P of P to P has a lifting to a $Mp(n)$ bundle $\widehat{\pi^*P} \to P$. We consider the associated bundle of symplectic spinors by taking an irreducible oscillator representation on $L^2(\mathbb{R}^n)$ of $Mp(n)$ and forming the associated real Hilbert bundle $\mathcal{W} = \widehat{\pi^*P} \times_{Mp(n)} L^2(\mathbb{R}^n)$ on P. $\mathcal{W}$ is a bundle gerbe module for L, called a metaplectic real bundle gerbe module. It is not hard to show that the possible metaplectic bundle gerbe modules for the metaplectic bundle gerbe $L \to P^{[2]}$ are parametrised by $H^1(M, \mathbb{Z}_2)$, i.e. the real line bundles on M,

6. KO-theory for real bundle gerbes

Given a real bundle gerbe (L, Y) we denote by $KO^0_{bg}(L)$ the Grothendieck group of the semi-group $\mathrm{Mod}_{\mathbb{R}}(L)$ and call this the KO group of the real bundle gerbe. We also define $KO^j_{bg}(L) = KO^0_{bg}(p_j^*L)$ where $p_j : M \times \mathbb{R}^j \to M$ is the projection and p_j^*L is the pullback real bundle gerbe over $M \times \mathbb{R}^j$. By proposition 3, one has:

Proposition 5. *A choice of stable isomorphism from L to J defines a canonical isomorphism $KO^j_{bg}(L) \simeq KO^j_{bg}(J)$ for all $j \geq 0$.*

Notice that the group $KO^\bullet_{bg}(L)$ depends only on the Dixmier-Douady class $d(L) \in H^2(M, \mathbb{Z}_2)$ and for any class $[H]$ in $H^2(M, \mathbb{Z}_2)$ we can define a real bundle gerbe L with $d(L) = [H]$ and hence the groups $KO^\bullet_{bg}(L)$. When we want to emphasise the dependence on $[H]$ we denote this by $KO^\bullet_{bg}(M, [H])$.

It is easy to deduce from the theory of real bundle gerbes various properties of this KO-theory:

Proposition 6. *Real bundle gerbe KO theory satisfies the following properties:*

1. *If (L, Y) is a trivial real bundle gerbe over M, then $KO^\bullet_{bg}(L) = KO^\bullet(M)$.*
2. *If (L, Y) is a real bundle gerbe over M, then $KO^\bullet_{bg}(L)$ is a module over $KO^0(M)$.*
3. *If (L, Y) and (J, X) are real bundle gerbes over M, then there is a homomorphism*
$$KO^i_{bg}(L) \otimes KO^j_{bg}(J) \to KO^{i+j}_{bg}(L \otimes J)$$
where $L \otimes J$ denotes the bundle gerbe over the fibre product of Y and X.
4. *If (L, Y) is a real bundle gerbe over M and $f : N \to M$ is a continuous map, there is a homomorphism*
$$KO^\bullet_{bg}(L) \to KO^\bullet_{bg}(f^*L)$$
where f^*L denotes the pullback bundle gerbe over N.

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The proofs of these statements are entirely analogous for the corresponding proposition 5.2 of [6]. There is another construction that associates to any class $[H]$ in $H^2(M, \mathbb{Z}_2)$, groups $KO^\bullet(M, [H])$ called the *twisted* KO group. Twisted KO-theory shares the same properties as those in proposition 6. In the next section we discuss twisted KO-theory and show that real bundle gerbe KO-theory and twisted KO-theory coincide.

7. Type-I D-brane charges in an H-flux, twisted KO-theory and real bundle gerbe modules

7.1 Twisted KO-theory

We recall the definition of twisted KO-theory [9]. Given a class $[H] \in H^2(M, \mathbb{Z}_2)$ choose a $PO(\mathcal{H})$ bundle Y whose class is $[H]$. It is known that that the group $Aut(\mathcal{K})$ of automorphisms of the C^*-algebra of real compact operators on $\mathcal{H}$ is equal to $PO(\mathcal{H})$, cf. [9]. We can form an associated algebra bundle,

$$\mathcal{E} = Y \times_{PO(\mathcal{H})} \mathcal{K}\,,$$

where $PO(\mathcal{H})$ acts on $\mathcal{K}$ via the adjoint action. Then the twisted KO-theory is by definition

$$KO^j(M, [H]) = KO_j(C_o(M, \mathcal{E}))$$

i.e. the KO-theory of the C^*-algebra of continuous sections vanishing at infinity of $\mathcal{E}$. We recall that

$$KO_j(C_o(M, \mathcal{E})) = KO_0(C_o(M, \mathcal{E}) \otimes C_o(\mathbb{R}^j)) \tag{7.1}$$

where $C_o(\mathbb{R}^j)$ denotes the C^*-algebra of continuous functions on $\mathbb{R}^j$ that vanish at infinity.

Now let $\mathcal{H}$ be a real infinite dimensional Hilbert space that is a $*$-module for the real Clifford algebra of $\mathbb{R}^{j-1}$, namely Cl_{j-1}. Recall that when Cl_{j-1} is simple, then this representation is unique up to equivalence, and when it is not simple, then it is the direct sum of two simple algebras. The assumption made is that each simple subalgebra is represented with infinite multiplicity on $\mathcal{H}$. Let $\widehat{\text{Fred}}$ denote the space of all skew-adjoint Fredholm operators on $\mathcal{H}$. For $k \geq 1$, let Fred_j denote the subspace of all skew-adjoint Fredholm operators $\widehat{\text{Fred}}$, that commute with the action of Cl_{j-1}. Then it has been shown in [14] that Fred_j is the classifying space for KO^j. One can form the associated bundle

$$Y(\text{Fred}_j) = Y \times_{PO(\mathcal{H})} \text{Fred}_j$$

where Fred_j is acted on by conjugation. Let $[M, Y(\text{Fred}_j)]$ denote the space of all homotopy classes of sections of $Y(\text{Fred}_j)$ then one has, cf. [9],

$$KO^j(M, [H]) = [M, Y(\text{Fred}_j)] = [Y, \text{Fred}_j]^{PO(\mathcal{H})}\,,$$

where the right hand side denotes the space of all homotopy classes of equivariant maps with the homotopies being by equivariant maps.

7.2 Real bundle gerbe KO-theory and twisted KO-theory

Here we will prove that real bundle gerbe KO-theory and twisted KO-theory are the same and indicate their relationship with equivariant KO-theory.

The Serre-Patterson theorem cf. [12] says that, given an element $[H] \in H^2(M, \mathbb{Z}_2)$, there is a principal $PO(n)$ bundle $X \to M$, with Dixmier-Douady invariant equal to $[H]$. We can define an action $\mathrm{SO}(n)$ on $\mathbb{R}^n \otimes \mathcal{H} = \mathcal{H}^n$ letting g act as $g \otimes 1$. This gives a representation $\rho_n \colon \mathrm{SO}(n) \to O(\mathcal{H}^n)$ and induces a $PO(\mathcal{H}^n)$ bundle with Dixmier-Douady class $[H]$. As $\mathcal{H}^n \simeq \mathcal{H}$ and all $PO(\mathcal{H})$ bundles are determined by their Dixmier-Douady class we can assume that this bundle is Y and contains X as a $\mathrm{SO}(n)$ reduction. Then we have

$$(Y \times \mathrm{Fred})/PO(\mathcal{H}) \cong (X \times \mathrm{Fred})/PO(n)\,,$$

so that

$$KO^0(M, [H]) = [Y, \mathrm{Fred}]^{PO(\mathcal{H})} \cong [X, \mathrm{Fred}]^{PO(n)}\,.$$

The lifting real bundle gerbe for $Y \to M$ pulls-back to become the lifting real bundle gerbe L for $X \to M$. We will now prove that $KO^0_{bg}(M, L) = KO^0(M, [H])$. Notice that this will prove the result also for *any* real bundle gerbe with torsion Dixmier-Douady class as we already know that real bundle gerbe K-theory depends only on the Dixmier-Douady class.

In the case where there is no twist, Atiyah and Janich showed that $KO(M) = [M, \mathrm{Fred}]$ and we will follow Atiyah's proof indicating just what needs to be modified to cover this equivariant case.

First we have have the following

Lemma 1. *If W is a finite dimensional subspace of $\mathbb{R}^n \otimes \mathcal{H}$ there is a finite co-dimensional subspace V of $\mathcal{H}$ such that $\mathbb{R}^n \otimes V \cap W = 0$.*

Proof. Let U be the image of V under the map $\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathcal{H} \to \mathcal{H}$ where we contract the two copies of $\mathbb{R}^n$ with the inner product. Then $V \subset \mathbb{R}^n \otimes U$. So take $W = U^\perp$. □

Using the compactness of X and the methods in Atiyah and Janich, cf. the appendix of [15], we can show that if $f \colon X \to \mathrm{Fred}(\mathbb{R}^n \otimes \mathcal{H})$ is continuous, then there is a subspace $V \subset \mathcal{H}$, of finite co-dimension, such that $\ker(f(x)) \cap \mathbb{R}^n \otimes V = 0$ for all $x \in X$ and $\bigcup_{x \in X} \mathcal{H}/f(x)(V) := \mathcal{H}/f(V)$ topologized as a quotient space of $X \times \mathcal{H}$, is a real vector bundle over X. Then $\mathcal{H}/V$ and $\mathcal{H}/f(V)$ also have an action of $\mathrm{SO}(n)$ in such a way as to make them real bundle gerbe modules. So we define

$$\mathrm{ind} \colon [X, \mathrm{Fred}(\mathbb{R}^n \otimes \mathcal{H})]_{\mathrm{SO}(n)} \to KO_{bg}(M, L)$$

by $\mathrm{ind}(f) = [\mathcal{H}/V - \mathcal{H}/f(V)]$. Again the methods of [15] will show that this index map is well-defined and a homomorphism.

As in [15] we can identify the kernel of ind as $[X, O(\mathbb{R}^n \otimes \mathcal{H})]_{\mathrm{SO}(n)}$ and use the result of Segal [16] which shows that $O(\mathbb{R}^n \otimes \mathcal{H})$ is contractible so ind is injective.

Finally we consider surjectivity. First we need from [17] the following

Proposition 7. *If $E \to X$ is a real bundle gerbe module for L then there is a representation $\mu\colon \mathrm{SO}(n) \to \mathrm{SO}(n)$ such that E is a sub-real bundle gerbe module of $\mathbb{R}^N \otimes X$.*

If $E \to X$ is a real bundle gerbe module then proposition 7 enables us to find a $\mathrm{SO}(n)$ equivariant map $\tilde{f}\colon X \to \mathrm{Fred}(\mathbb{R}^N \otimes X)$ whose index is E. The action of $\mathrm{SO}(n)$ used here on $\mathbb{R}^N \otimes X$ is that induced from the representation μ. To prove surjectivity of the index map it suffices to find a map $f\colon X \to \mathrm{Fred}(\mathbb{R}^n \otimes X)$ whose index is E. Then if $E - F$ is a class in $KO_{bg}(M, L)$ we can apply a similar technique to obtain a map whose index is $-F$ and combine these to get a map whose index is $E - F$ and we are done.

To construct f we proceed as follows. We have a representation $\rho_n\colon \mathrm{SO}(n) \to \mathbb{R}^n \otimes \mathcal{H}$ and a representation $\rho_N\colon \mathrm{SO}(n) \to \mathbb{R}^N \otimes \mathcal{H}$. These can be used to induce a $PO(\mathbb{R}^n \otimes \mathcal{H})$ bundle and a $PO(\mathbb{R}^N \otimes \mathcal{H})$ bundle, both with Dixmier-Douady class $[H]$. So they must be isomorphic. We need the precise form of this isomorphism. Choose an isomorphism $\phi\colon \mathbb{R}^n \otimes \mathcal{H} \to \mathbb{R}^N \otimes \mathcal{H}$. This induces an isomorphism $O(\mathbb{R}^n \otimes \mathcal{H}) \to O(\mathbb{R}^N \otimes \mathcal{H})$ given by $u \mapsto \phi u \phi^{-1}$ which we will denote by $\phi[u]$ for convenience. There is a similar identification $\mathrm{Fred}(\mathbb{R}^n \otimes \mathcal{H}) \to \mathrm{Fred}(\mathbb{R}^N \otimes \mathcal{H})$. The two PU bundles are given by $X \times_{\rho_n} O(\mathbb{R}^n \otimes \mathcal{H})$ and $X \times_{\rho_N} O(\mathbb{R}^N \otimes \mathcal{H})$ and consist of cosets $[x, u] = [xg, \rho_n^{-1}(g)u]$ and $[x, u] = [xg, \rho_N^{-1}(g)u]$ respectively. The action of $O(\mathbb{R}^n \otimes \mathcal{H})$ is $[x, u]v = [x, uv]$ and similarly for $O(\mathbb{R}^N \otimes \mathcal{H})$. Because these are isomorphic bundles there must be a bundle map

$$\phi\colon X \times_{\rho_n} O(\mathbb{R}^n \otimes \mathcal{H}) \to \times_{\rho_N} O(\mathbb{R}^N \otimes \mathcal{H})$$

satisfying $\phi([x, u]v) = \phi([x, u])\phi[v]$ and hence $\phi([x, u]) = \phi([x, 1])\phi[u]$. Define $\alpha\colon X \to O(\mathbb{R}^N \otimes \mathcal{H})$ by requiring that $\phi([x, 1]) = [x, \alpha(x)]$. Then for $g \in \mathrm{SO}(n)$, a calculation analogous to [6] shows that

$$\alpha(xg) = \rho_N(g)^{-1}\alpha(x)\phi[(\rho_n(g)]. \tag{7.2}$$

We can now define $f\colon X \to \mathrm{Fred}(\mathbb{R}^n \otimes X)$ by

$$f(x) = \alpha(x)\phi^{-1}[\tilde{f}(x)]$$

and it is straightforward to see that this is $\mathrm{SO}(n)$ equivariant by applying the equation (7.2). It is clear that $\alpha(x)$ and ϕ can be used to establish an isomorphism between $\ker(f)$ and $\ker(\tilde{f})$ and hence between $\ker(f)$ and E.

This proves

Proposition 8. *If L is a real bundle gerbe over M with Dixmier-Douady class $[H] \in H^2(M, \mathbb{Z}_2)$, then $KO^0_{bg}(M, L) = KO^0(M, [H])$.*

The lifting real bundle gerbe for $Y \to M$ pulls-back to become the lifting real bundle gerbe for $X \to M$. A real bundle gerbe module for this is a bundle $E \to X$ with a $\mathrm{SO}(n)$ action covering the action of $PO(n)$ on X. This $\mathrm{SO}(n)$ action has to have the property that the center $\mathbb{Z}_2 \subset \mathrm{SO}(n)$ acts on the fibres of E by scalar multiplication. Considered from this perspective we see that we are in the context of equivariant KO theory [17]. Notice that by projecting to $PO(n)$ we can make $\mathrm{SO}(n)$ act on X. Of course this action is not free, the center $\mathbb{Z}_2$ is the isotropy subgroup at every point. The equivariant KO theory $KO_{\mathrm{SO}(n)}(X)$ is the KO theory formed from vector bundles on X which have an $\mathrm{SO}(n)$ action covering the action on X. We need a subset of such bundles with a particular action. To understand this note that because the center $\mathbb{Z}_2 \subset \mathrm{SO}(n)$ is the isotropy subgroup for the $\mathrm{SO}(n)$ action on X it must act on the fibres of E and hence define a representation of $\mathbb{Z}_2$ on $\mathbb{R}^r$ if the bundle E has rank r. This defines an element of $R(\mathbb{Z}_2)$, the representation ring of $\mathbb{Z}_2$, and to be a real bundle gerbe module this representation must be scalar multiplication on $\mathbb{R}^r$. In terms of equivariant KO theory we can consider the map which is restriction to a fibre of $X \to M$ and then we have

$$K_{\mathrm{SO}(n)}(X) \to K_{\mathrm{SO}(n)}(\mathrm{SO}(n)/\mathbb{Z}_2) = R(\mathbb{Z}_2).$$

$KO_{bg}(M, L)$ is the pre-image under this map of the representation of $\mathbb{Z}_2$ on $\mathbb{R}^n$ by scalar multiplication.

Defining $KO^j_{bg}(M, L) = KO^0_{bg,c}(M \times \mathbb{R}^j, L)$, where $KO^0_{bg,c}$ denotes compactly supported bundle gerbe K-theory, we deduce from proposition 8 that

Proposition 9. *If L is a real bundle gerbe over M with Dixmier-Douady class $[H] \in H^2(M, \mathbb{Z}_2)$, then $KO^j_{bg}(M, L) = KO^j(M, [H])$ for all $j \geq 0$.*

8. Infinite dimensional model for twisted KO-theory

We have seen that twisted KO-theory can be defined using finite rank real bundle gerbes. But sometimes it is useful to get an infinite dimensional generalisation such as to allow real bundle gerbe modules which are infinite dimensional real Hilbert bundles. In that case the induced projective bundle on M is a $PO(\mathcal{H})$ bundle for $\mathcal{H}$ an infinite dimensional real Hilbert bundle and it is well known that there is only one such bundle for a given Dixmier-Douady class and hence proposition 4 implies that

Proposition 10. *Every real bundle gerbe admits exactly one real bundle gerbe module which is a bundle of infinite dimensional Hilbert spaces with structure group $O(\mathcal{H})$.*

In particular if E and F are Hilbert real bundle gerbe modules then $E = F$ so that the class $E = F$ in the induced KO group is zero. So the KO group is zero.

In the remainder of this section we discuss another approach to twisted cohomology where the structure group of the real bundle gerbe module is the group $O_{\mathcal{K}}$, the subgroup of $O(\mathcal{H})$ of orthogonal operators which differ from the identity by a compact operator (here $\mathcal{K}$ denotes the compact operators on $\mathcal{H}$). To see how this arises notice that in Rosenberg's definition [9] we can replace Fred by a homotopy equivalent space. For our purposes we choose $BO_{\mathcal{K}} \times \mathbb{Z}$. This can be done in a $PO(\mathcal{H})$ equivariant fashion as follows. For $BO_{\mathcal{K}}$ we could choose the connected component of the identity of the invertibles in the real Calkin algebra $B(\mathcal{H})/\mathcal{K}$ which is homotopy equivalent in a $PO(\mathcal{H})$ equivariant way to the real Fredholms of index zero under the quotient map $\pi : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}$. Note however that the identity component of the invertibles in the Calkin algebra is just $GL(\mathcal{H})/GL_{\mathcal{K}}$ where

$GL(\mathcal{H})$ denotes the invertible operators on $\mathcal{H}$ and $GL_{\mathcal{K}}$ are the invertibles differing from the identity by a compact. Thus we can take $BO_{\mathcal{K}}$ to be $GL(\mathcal{H})/GL_{\mathcal{K}}$ and this choice is $PO(\mathcal{H})$ equivariant. We could equally well take $BO_{\mathcal{K}} = O(\mathcal{H})/O_{\mathcal{K}}$.

As $O(\mathcal{H})$ acts on $O_{\mathcal{K}}$ by conjugation there is a semi-direct product

$$O_{\mathcal{K}} \to O_{\mathcal{K}} \rtimes PO(\mathcal{H}) \to PO(\mathcal{H}) .$$

Note that this means that any $O_{\mathcal{K}} \rtimes PO(\mathcal{H})$ bundle over M induces a $PO(\mathcal{H})$ bundle and hence a class in $H^2(M, \mathbb{Z}_2)$. If $R \to Y$ is a $O_{\mathcal{K}}$ bundle we call it $PO(\mathcal{H})$ covariant if there is an action of $PO(\mathcal{H})$ on the right of R covering the action on Y such that $(rg)[u] = r[u]u^{-1}gu$ for any $r \in R$, $[u] \in PO(\mathcal{H})$ and $g \in O_{\mathcal{K}}$. Here $[u]$ is the projective class of some $u \in O(\mathcal{H})$.

Because $BO_{\mathcal{K}}$ is homotopy equivalent to only the connected component of index 0 of Fred it is convenient to work with reduced twisted KO theory, $\widetilde{KO}(M, [H])$, defined by

$$\widetilde{KO}(M, [H]) = [M, Y(BO_{\mathcal{K}})] .$$

We have

Proposition 11. *Given a $PO(\mathcal{H})$ bundle $Y \to M$ with Dixmier-Douady class $[H] \in H^2(M, \mathbb{Z}_2)$ the following are equivalent, with a proof analogous to that given in [6].*

1. $\widetilde{KO}(M, [H])$
2. *space of homotopy classes of sections of $Y \times_{PO(\mathcal{H})} BO_{\mathcal{K}}$*
3. *space of homotopy classes of $PO(\mathcal{H})$ equivariant maps from Y to $BO_{\mathcal{K}}$*
4. *space of isomorphism classes of $PO(\mathcal{H})$ covariant $O_{\mathcal{K}}$ bundles on Y, and*
5. *space of isomorphism classes of $O_{\mathcal{K}} \rtimes PO(\mathcal{H})$ bundles on M whose projection to a $PO(\mathcal{H})$ bundle has class $[H]$.*

8.1 $O_{\mathcal{K}}$ real bundle gerbe modules

Given a $PO(\mathcal{H})$ covariant $O_{\mathcal{K}}$ bundle R over Y we can define the associated bundle

$$E = R \times_{O_{\mathcal{K}}} \mathcal{H} \to Y . \tag{8.1}$$

We claim that this is a real bundle gerbe module for the lifting real bundle gerbe P. Let $[r, v] \in E_{y_1}$ be a $O_{\mathcal{K}}$ equivalence class where $r \in R_{y_1}$, the fibre of R over $y_1 \in Y$ and $v \in \mathcal{H}$. Let $u \in L_{y_1 y_2}$ be an element of the lifting bundle gerbe. Then, by definition, $u \in O(\mathcal{H})$ and $y_1[u] = y_2$. We define the action of u by $[r, v]u = [r[u], u^{-1}v]$. It is straightforward to check that this is well defined. Hence we have associated to any $PO(\mathcal{H})$ covariant $O_{\mathcal{K}}$ bundle R on Y a module for the lifting real bundle gerbe.

The inverse construction is also possible if the real bundle gerbe module is a $O_{\mathcal{K}}$ real bundle gerbe module which we now define. Let $E \to Y$ be a Hilbert bundle with structure group $O_{\mathcal{K}}$. We recall what it means for a Hilbert bundle to have structure group $O_{\mathcal{K}}$. To any Hilbert bundle there is associated a $O(\mathcal{H})$ bundle $O(E)$ whose fibre, $O(E)_y$, at y is all unitary isomorphisms $f\colon \mathcal{H} \to E_y$. If $u \in O(\mathcal{H})$ it acts on $O(\mathcal{H})_y$ by $fu = f \circ u$ and hence $O(E)$ is a principal $O(\mathcal{H})$ bundle. For E to have structure group $O_{\mathcal{K}}$ means that we have a reduction of $O(E)$ to an $O_{\mathcal{K}}$ bundle $R \subset O(E)$. Each $R_y \subset O(E)_y$ is an orbit under $O_{\mathcal{K}}$, that is R is a principal $O_{\mathcal{K}}$ bundle.

For E to be an $O_{\mathcal{K}}$ real bundle gerbe module we need to define an action of the real bundle gerbe on it. By comparing with the action on the bundle E defined in (8.1) we see that we need to make the following definition. If $u \in O(\mathcal{H})$ such that $y_1[u] = y_2$ then $u \in L_{(y_1,y_2)}$ where $L \to Y^{[2]}$ is the lifting real bundle gerbe so if $f \in R_{y_1}$ then $ufu^{-1} \in O(E)_{y_2}$. We require that $ufu^{-1} \in R_{y_2}$. So a lifting real bundle gerbe module which is an $O_{\mathcal{K}}$ Hilbert bundle and satisfies this condition we call an $O_{\mathcal{K}}$ real bundle gerbe module. By construction we have that the associated R is an $O_{\mathcal{K}}$ bundle over Y on which $PO(\mathcal{H})$ acts. Let us denote by $\mathrm{Mod}_{O_{\mathcal{K}}}(M,[H])$ the semi-group of all $O_{\mathcal{K}}$ real bundle gerbe modules for the lifting real bundle gerbe of the $PO(\mathcal{H})$ bundle with Dixmier-Douady class $[H] \in H^2(M;\mathbb{Z}_2)$. As any two $PO(\mathcal{H})$ bundles with the same Dixmier-Douady class are isomorphic we see that $\mathrm{Mod}_{O_{\mathcal{K}}}(M,[H])$ depends only on $[H]$.

We have now proved

Proposition 12. *If (L,Y) is the lifting real bundle gerbe for a $PO(\mathcal{H})$ bundle with Dixmier-Douady class $[H]$*

$$\widetilde{K}O(M,[H]) = \mathrm{Mod}_{O_{\mathcal{K}}}(M,[H])\,.$$

If L_1 and L_2 are two $PO(\mathcal{H})$ covariant $O_{\mathcal{K}}$ bundles on Y note that $L_1 \times L_2$ is an $O_{\mathcal{K}} \times O_{\mathcal{K}}$ bundle. Choose an isomorphism $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ which induces an isomorphism $O_{\mathcal{K}} \times O_{\mathcal{K}} \to O_{\mathcal{K}}$ and hence defines a new $PO(\mathcal{H})$ covariant bundle $L_1 \otimes L_2$. It is straightfoward to check that

$$(L_1 \otimes L_2)(\mathcal{H}) \simeq L_1(\mathcal{H}) \times L_2(\mathcal{H})\,.$$

This makes $\mathrm{Mod}_{O_{\mathcal{K}}}(M,[H])$ a semi-group and the map $\widetilde{K}O(M,[H]) = \mathrm{Mod}_{O_{\mathcal{K}}}(M,[H])$ is a semi-group isomorphism. Note that with our definition $BO_{\mathcal{K}}$ is a group. Moreover the space of all equivariant maps $Y \to BO_{\mathcal{K}}$ is a group as well. To see this notice that if f and g are equivariant maps and we multiply pointwise then for $y \in Y$ and $[u] \in PO(\mathcal{H})$ we have

$$\begin{aligned}(fg)(y[u]) &= f(y[u])g(y[u])\\ &= (u^{-1}f(y)u)(u^{-1}g(y)u)\\ &= u^{-1}(fg)(y)u\end{aligned}$$

and if f^{-1} is the pointwise inverse then $f^{-1}(y[u]) = \left(u^{-1}f(y)u\right)^{-1} = u^{-1}f^{-1}(y)u$. This induces a group structure on $\mathrm{Mod}_{O_{\mathcal{K}}}(M,[H])$. We have already noted that it is a semi-group but this implies more, for every $O_{\mathcal{K}}$ bundle gerbe module E there is an $O_{\mathcal{K}}$ real bundle gerbe module E^{-1} such that $E \oplus E^{-1}$ is the trivial $O_{\mathcal{K}}$ real bundle gerbe.

Note 1. The group $O_{\mathcal{K}}$ used here could be replaced by any other group to which it is homotopy equivalent by a homotopy equivalence preserving the $PO(\mathcal{H})$ action. In particular we could consider O_1, the subgroup of $O(\mathcal{H})$ consisting of orthogonal operators which differ from the identity by a trace class operator — it is a result of Palais and Schwartz that O_1 is homotopy equivalent to $O_{\mathcal{K}}$, cf. [18]. In section 9 we show that the computation in section 6 of real bundle gerbe characteristic classes generalizes, with some modifications, to O_1 real bundle gerbe modules.

8.2 Local description of $O_{\mathcal{K}}$ real bundle gerbe modules

Let $\{U_i\}_{i\in I}$ be a good cover of M and let $U_{ij\ldots k} = U_i \cap U_j \cap \ldots \cap U_k$. The trivial bundle has a sections s_i which are related by

$$s_i = s_j[u_{ji}]$$

where $[u_{ji}]\colon U_{ij} \to PO(\mathcal{H})$ for some $u_{ji}\colon U_{ij} \to O(\mathcal{H})$ where $u_{ij}u_{jk}u_{ki} = g_{ijk}1$ where 1 is the identity operator and the g_{ijk} are non-zero scalars.

Over each of the $s_i(U_i)$ are sections σ_i of the $O_{\mathcal{K}}$ bundle R. We can compare σ_i and $\sigma_j[u_{ji}]$ so that

$$\sigma_i = \sigma_j[u_{ji}]g_{ji}$$

where $g_{ij}\colon U_{ij} \to O_{\mathcal{K}}$. These satisfy

$$g_{ki} = ([u_{ji}^{-1}]g_{kj}[u_{ji}])g_{ji}\,. \tag{8.2}$$

If $Y_i = \pi^{-1}(U_i)$ one can define a section of R over all of Y_i by $\hat{\sigma}_i(s_i[u]) = \sigma_i[u]$. The transition functions for these are $\hat{g}_{ij}$ where $\hat{g}_{ij}(s_j[u]) = [u^{-1}]g_{ji}[u]$ and the identity (8.2) is equivalent to $\hat{g}_{ki} = \hat{g}_{kj}\hat{g}_{ji}$.

9. Continuous trace real C^*-algebras

There is a well-known construction of a continuous trace real C^* algebra from a groupoid [19, 9]. This can be used to construct a C^* algebra from real bundle gerbes as follows. If the fibres of $Y \to M$ have an appropriate Haar measure and we can define a product on two sections $f, g\colon Y^{[2]} \to P$ by

$$(fg)(y_1, y_2) = \int f(y_1, y)g(y, y_2)dy$$

where in the integrand we use the real bundle gerbe product so that $f(y_1, y)g(y, y_2) \in L_{(y_1,y_2)}$. Closing this space of sections in the operator norm topology gives a continuous trace real C^* algebra with spectrum M and Dixmier-Douady class the Dixmier-Douady class of (L, Y). Some constructions in the theory of real C^* algebras become easy from this perspective. For example if A is continuous trace real C^*-algebra with spectrum X and $f\colon Y \to X$ is a continuous map, then there is continuous trace real C^*-algebra $f^{-1}(A)$ with spectrum Y. This is just the pullback of real bundle gerbes. Then the KO_{bg}-theory is just the KO-theory of the associated continuous trace real C^*-algebra. This works as long as we take finite dimensional real bundle gerbes, which we know represent all classes in $H^2(M, \mathbb{Z}_2)$.

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10. Type-I D-brane charges ⇒ type-II D-brane charges

Here we discuss the natural complexification map from twisted KO-theory to twisted K-theory, which can be used to define the twisted Chern character homomorphism in twisted KO-theory. This homomorphism is analysed in some detail, together with examples.

10.1 The complexification homomorphism

We use the well known and fundamental fact that the complexification of a real vector vector bundle is a complex vector bundle that is isomorphic to its own conjugate vector bundle., cf. [15]. We begin with a real bundle gerbe (L, Y) over M, where $\pi : Y \to M$ is a locally trivial fibre bundle and $L \to Y^{[2]}$ is a real line bundle satisfying equation (2.1). The complexification of the real bundle (L, Y) determines the bundle gerbe $(L \otimes \mathbb{C}, Y)$ over M which is automatically isomorphic to $(\overline{L \otimes \mathbb{C}}, Y)$. If the Dixmier-Douady invariant of (L, Y) is $d(L) \in H^2(M, \mathbb{Z}_2)$, then it is easy to see that the Dixmier-Douady invariant of $(L \otimes \mathbb{C}, Y)$ is $d(L \otimes \mathbb{C}) \in H^3(M, \mathbb{Z})$, where $d(L \otimes \mathbb{C}) = \beta(d(L))$ and $\beta : H^2(M, \mathbb{Z}_2) \to H^3(M, \mathbb{Z})$ is the Bockstein homomorphism. Observe also that $d(L \otimes \mathbb{C}) = d(\overline{L \otimes \mathbb{C}}) = -d(L \otimes \mathbb{C})$. The complexification map is compatible with stable isomorphism of real and complex bundle gerbes, and therefore defines a homomorphism from stable equivalence classes of real bundle gerbes and stable equivalence classes of bundle gerbes.

Now let (L, Y) be a real bundle gerbe over a manifold M and let $E \to Y$ be a finite dimensional real bundle gerbe module. Then it is straightforward to see that the complexification $E \otimes \mathbb{C} \to Y$ is a bundle gerbe module for the bundle gerbe $(L \otimes \mathbb{C}, Y)$ over M. This map is compatible with real and complex bundle gerbe stable equivalences, and isomorphism of real and complex bundle gerbe modules, therefore defining a well defined homomorphism,

$$\otimes\, \mathbb{C} : KO^0(M, d(L)) \to K^0(M, d(L \otimes \mathbb{C})) \tag{10.1}$$

where we recall that $d(L \otimes \mathbb{C}) = \beta(d(L))$ as above.

We can use this homomorphism to define the twisted Chern character,

$$ch_{d(L)} : KO^0(M, d(L)) \to H^{even}(M, \mathbb{R}) \tag{10.2}$$

by $ch_{d(L)}(E) := ch_{d(L\otimes\mathbb{C})}(E\otimes\mathbb{C})$, where we have used the twisted chern character as defined in [6]. Explicitly, it is determined by the property

$$\pi^*(ch_{d(L)}(E)) = e^{c_1(L\otimes\mathbb{C})} ch(E \otimes \mathbb{C})\,.$$

It is a homomorphism with the usual properties, namely,

1. $ch_{d(L)}$ is natural with respect to pullbacks,
2. $ch_{d(L)}$ respects the $KO^0(M)$-module structure of $KO^0(M, d(L))$,
3. $ch_{d(L)}$ reduces to the ordinary Chern character in the untwisted case when $d(L) = 0$.

Since $E \otimes \mathbb{C} \to Y$ is isomorphic to $\overline{E \otimes \mathbb{C}} \to Y$ for every real bundle gerbe module $E \to Y$, we see that

$$\pi^*(ch_{d(L)}(E)) = e^{c_1(L\otimes\mathbb{C})}ch(E \otimes \mathbb{C}) = e^{c_1(\overline{L\otimes\mathbb{C}})}ch(\overline{E \otimes \mathbb{C}})$$

from which one deduces that all of the components of the twisted Chern character of degree $4j+2$ vanish. Therefore,

$$ch_{d(L)} : KO^0(M, d(L)) \to H^{4\bullet}(M, \mathbb{R}) \tag{10.3}$$

and is an isomorphism over the reals.

10.2 Remarks on the spectral sequence relevant to calculations

We will now briefly discuss some calculations of twisted KO-groups of compact surfaces, in terms of the untwisted KO-groups. The first non-trivial differential in the spectral sequence as described in [9] is $d_2 = H + Sq^2$, where Sq^2 denotes the second Steenrod square, which vanishes for compact surfaces. Therefore $d_2 = H$. Again for compact surfaces, there are no other non-trivial differentials. It follows that $KO^i(\Sigma, H) \cong KO^i(\Sigma)$ whenever $i \neq -2, -3$, since $KO^{-i}(pt)$ is equal to $\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0$ when i ranges from 0 to 7. Now let Σ be a compact surface and $[H] \in H^2(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2$ be nontrivial. Then following the spectral sequence, one obtains $KO^{-2}(\Sigma, H) \cong KO^{-2}(\Sigma)/\mathbb{Z}_2$ and also $KO^{-3}(\Sigma, H) \cong KO^{-3}(\Sigma)/\mathbb{Z}_2$.

11. Concluding remarks

In this paper we have developed the foundations of real bundle gerbes and the geometric interpretation of twisted KO-theory as the KO-theory of real bundle gerbe modules, as it may prove useful in the geometric understanding of D-brane charges and fields in type-I string theory. We have also studied the fundamental properties of twisted KO-theory including the Chern character as well as its relation to twisted K-theory etc.

In some work in progress, we study the real index bundle gerbe, called the pfaffian gerbe, that is associated to a family of skew adjoint real Dirac type operators, together with a natural metric on it as well as a compatible connection and curving for it. We also relate it to the (complex) index bundle gerbe under the complexification homomorphism. This serves as one of the motivations for this paper.

An interesting open question that remains unanswered is the following. A missing Chern-Weil or geometric interpretation of the Dixmier-Douady invariant of real bundle gerbes when the group $\mathrm{Ext}(H_1(M, \mathbb{Z}), \mathbb{Z}_2)$ does not vanish: this can happen whenever the torsion subgroup of $H_1(M, \mathbb{Z})$, has 2-torsion elements as discussed in section 4, eq. (4.1).

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